Online Influence Maximization under Linear Threshold Model

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Motivation

- Online influence maximization (OIM)
  - A sequential decision-making problem
  - Selects a set of users and provides them with free products
  - Receives feedback from the information diffusion process
- Common influence propagation models: independent cascade (IC) model, linear threshold (LT) model
- Existing works mainly focus on the IC model with edge-level feedback
  - The IC model assumes the information spreads through each edge independently.
  - The edge-level feedback means that the learner could observe the influence status of each edge once its start node is influenced.
- The LT model characterizes the herd behavior that often occurs in real information diffusion process, that with more active in-neighbors, a user becomes much more likely to be influenced.

Setting

- Graph $G=(V,E)$: V is the set of users (nodes) and E is the set of relationships (edges)
  - Each edge $e_{u,v}\in E$ is associated with a weight $w(e_{u,v})$ representing the influence ability of u on v
  - $n=|V|, m=|E|, D$ are node number, edge number and the diameter respectively, where the diameter of the graph is defined as the maximum (directed) distance between the pair of nodes in any connected component.
- The diffusion process starting from seed set S under LT model
  - Each node is assigned with a threshold θ_v, which is independently uniformly drawn from $[0,1]$ and characterizes the susceptibility level of v.
  - Let S_τ be the set of activated nodes by the end of time τ.
    - $*$ At time $\tau=0$, only nodes in the seed set are activated: $S_0=S$.
    - $*$ At time $\tau+1$ with $\tau\geq 0$, for any node $v\notin S_\tau$ that has not been activated yet, it will be activated if the aggregated influence of its active in-neighbors exceeds its threshold: $\sum_{u\in N(v)\cap S_\tau} w(e_{u,v})\geq\theta_v$.
    - $*$ Such diffusion process will last at most D time steps.
  - The size of the influenced nodes: $r(S,w,\theta)=|S_D|$
  - Let $r(S,w)=\mathbb{E}[r(S,w,\theta)]$ be the *influence spread* of seed set S where the expectation is taken over all random variables θ_v's; $\mathrm{Opt}_w=\max_{|S|\leq K} r(S,w)$ be the maximum influence spread under weight vector w with limited size K
- Some notations in the diffusion process
  - $\tau_1(v):=\min_\tau\{\tau=0,\dots,D: N(v)\cap S_\tau\neq\emptyset\}$ is the earliest time step when node v has active neighbors, set $\tau_1(v)=D+1$ if node v has no active in-neighbor until the diffusion ends.
  - $\tau_2(v):=\left\{\tau=0,\dots,D:\chi(E_{\tau-2}(v))^\top w_v<\theta_v\leq\chi(E_{\tau-1}(v))^\top w_v\right\}$ is the time step that node v is influenced, set $\tau_2(v)=D+1$ if node v is finally not influenced after the information diffusion ends.
  - $E_\tau(v):=\left\{e_{u,v}:u\in N(v)\cap S_\tau\right\}$ is the set of incoming edges associated with active in-neighbors of v at time step τ.
- The online IM (OIM) problem, in each round t:
  - The learner chooses a set of nodes S_t with limited size K
  - The learner observes (full) node-level feedback $S_{t,0},S_{t,1},\dots,S_{t,D}$, where $S_{t,i}$ represents the set of active nodes by time step $i\in[D]$ in the diffusion process in this round.
  - The learner updates its knowledge on unknown weights using the observed feedback, which helps the seed set selection in the next round.
  - The goal of the learner is to minimize the expected cumulative η-scaled regret

$$R(T)=\mathbb{E}\left[\sum_{t=1}^T R_t\right]=\mathbb{E}\left[\eta\cdot T\cdot\mathrm{Opt}_w-\sum_{t=1}^T r(S_t,w)\right].$$

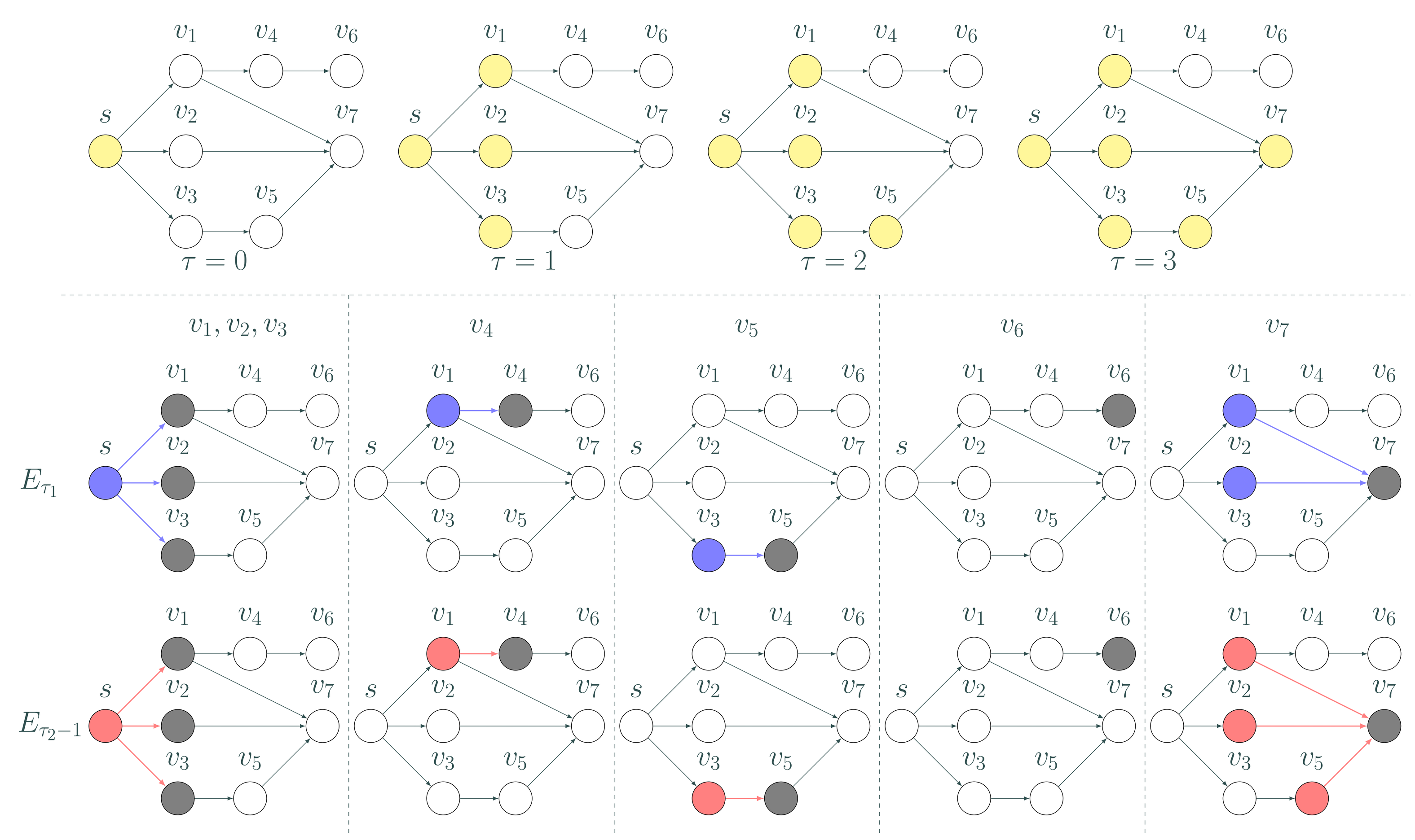

Figure 1: An example of diffusion process starting from $S=\{s\}$ under LT. The upper part describes an influence diffusion process where yellow nodes represent influenced nodes by the current time. The lower part describes what E_{τ_1},E_{τ_2-1} are where we use blue (red) color to represent the edges and the associated active in-neighbors in E_{τ_1} (E_{τ_2-1}, respectively) for the objective black node.

Algorithm

`LT-LinUCB` **Algorithm:**

Input: Graph $G=(V,E)$; seed set cardinality K; exploration parameter $\rho_{t,v}>0$ for any t,v; offline oracle `PairOracle`
Initialize: $M_{0,v}\leftarrow I\in\mathbb{R}^{|N(v)|\times|N(v)|}$, $b_{0,v}\leftarrow 0\in\mathbb{R}^{|N(v)|\times 1}$, $\hat{w}_{0,v}\leftarrow 0\in\mathbb{R}^{|N(v)|\times 1}$ for any node $v\in V$
for $t=1,2,3,\dots$
  Compute the confidence ellipsoid $\mathcal{C}_{t,v}=\left\{w'_v\in[0,1]^{|N(v)|\times 1}:\left\|w'_v-\hat{w}_{t-1,v}\right\|_{M_{t-1,v}}\leq\rho_{t,v}\right\}$ for any node $v\in V$
  Compute the pair (S_t,w_t) by `PairOracle` with confidence set $\mathcal{C}_t=\left\{\mathcal{C}_{t,v}\right\}_{v\in V}$ and seed set cardinality K
  Select the seed set S_t and observe the feedback
  //Update
  for node $v\in V$
    Initialize $A_{t,v}\leftarrow 0\in\mathbb{R}^{|N(v)|\times 1}$, $y_{t,v}\leftarrow 0\in\mathbb{R}$
    Uniformly randomly choose $\tau\in\left\{\tau':\tau_{t,1}(v)\leq\tau'\leq\tau_{t,2}(v)-1\right\}$
    if v is influenced and $\tau=\tau_{t,2}(v)-1$
      $A_{t,v}=\chi\left(E_{t,\tau}(v)\right),\quad y_{t,v}=1$
    else if $\tau=\tau_1(v),\dots,\tau_2(v)-2$ or $\tau=\tau_2(v)-1$ but v is not influenced
      $A_{t,v}=\chi\left(E_{t,\tau}(v)\right),\quad y_{t,v}=0$
    $M_{t,v}\leftarrow M_{t-1,v}+A_{t,v}A_{t,v}^\top,\quad b_{t,v}\leftarrow b_{t-1,v}+y_{t,v}A_{t,v},\quad \hat{w}_{t,v}=M_{t,v}^{-1}b_{t,v}$

Analysis

- For the seed set S, define the set of all nodes related to a node v, $V_{S,v}$, to be the set of nodes that are on any path from S to v in graph G.
- For seed set S and node $u\in V\setminus S$, define $N_{S,u}:=\sum_{v\in V\setminus S}\mathbb{1}\{u\in V_{S,v}\}\leq n-K$ to be the number of nodes that u is relevant to.
- Then for the vector $N_S=(N_{S,u})_{u\in V}$, define the upper bound of its L^2-norm over all feasible seed sets

$$\gamma(G):=\max_{S\in\mathcal{A}}\sqrt{\sum_{u\in V}N_{S,u}^2}\leq(n-K)\sqrt{n}=O(n^{3/2}),$$

which is a constant related to the graph.

Theorem 1 (GOM bounded smoothness). *For any two weight vectors $w,w'\in[0,1]^m$ with $\sum_{u\in N(v)}w(e_{u,v})\leq 1$, the difference of their influence spread for any seed set S can be bounded as*

$$\left|r(S,w')-r(S,w)\right|\leq\mathbb{E}\left[\sum_{v\in V\setminus S}\sum_{u\in V_{S,v}}\sum_{\tau=\tau_1(u)}^{\tau_2(u)-1}\left|\sum_{e\in E_\tau(u)}(w'(e)-w(e))\right|\right]$$

where the definitions of $\tau_1(u),\tau_2(u)$ and $E_\tau(u)$ are all under weight vector w, and the expectation is taken over the randomness of the thresholds on nodes.

Theorem 2 (Upper Bound). *Suppose the* `LT-LinUCB` *runs with an (α,β)-approximation* `PairOracle` *and parameter $\rho_{t,v}=\rho_t=\sqrt{n\log(1+tn)+2\log\frac{1}{\delta}}+\sqrt{n}$ for any node $v\in V$. Then the $\alpha\beta$-scaled regret satisfies*

$$R(T)\leq 2\rho_T\gamma(G)D\sqrt{mnT\log(1+T)/\log(1+n)}+n\delta\cdot T(n-k).$$

When $\delta=1/(n\sqrt{T})$, $R(T)\leq C\cdot\gamma(G)\,Dn\sqrt{mT}\log(T)$ for some universal constant C.

Conclusions

- Formulate the problem of OIM under LT model with node-level feedback and design how to distill effective information from observations.
- Prove a novel GOM bounded smoothness property for the spread function.
- Propose `LT-LinUCB` algorithm with rigorous theoretical analysis and show a competitive regret bound of $O(\mathrm{poly}(m)\sqrt{T}\ln(T))$.
- Design `OIM-ETC` algorithm with theoretical analysis on its distribution-dependent and distribution-independent regret bounds.
  - Efficient, applies to both LT and IC models, and has less requirements on feedback and offline computation.
- Future work: The OIM problem under IC model with node-level feedback; Applying Thompson sampling to influence maximization

References

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